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On Lie Subalgebras of Associative PI-Algebras

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1. PRELIMINARIES

1.1. Following V. N. Latyšev [9] a Lie algebra L over a commutative ring k is called an SPI-algebra if there exists an associative PI-algebra A over k and a k-linear embedding $i: L \to A$ such that for all $x, y \in L$

$$i([x, y]) = i(x)\, i(y) - i(y)\, i(x).$$

In other words, if we consider A as a Lie algebra under $[x, y] = xy - yx$, then L is isomorphic to a Lie subalgebra of A.

1.2. In [9] it was shown that finite dimensional and nilpotent Lie algebras are SPI. Latyšev also proved that every nilpotent-by-abelian Lie algebra L is SPI. He showed that if a soluble Lie algebra L is SPI then it is a nilpotent-by-abelian extension of a locally nilpotent ideal, that is, for some m, $[L, L]^m$ is locally nilpotent. He deduced from this that no free nonabelian soluble Lie algebra L of solubility length $\geqslant 4$ over a field of characteristic 0 is SPI. It was conjectured that the homomorphic image of SPI-algebra is not always SPI. Latyšev asked: What are the necessary and sufficient conditions in Lie algebra language which ensure that L is SPI?

1.3. None of these questions is completely answered in the present paper; but we answer them in the case of finitely generated virtually soluble Lie algebras (4.6). We give a method of constructing a new SPI algebra from two given ones (4.2). Latyšev's method for nilpotent-by-abelian algebra is slightly extended to yield information in the case of fields of nonzero characteristic (2.2). We show in 1.8 that a simple SPI-algebra is finite dimensional over its centroid.

1.4. Our results depend on standard theorems from the theory of associative PI-algebras. Most of them can be found in Procesi [11]. Some

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arguments in Section 3 are borrowed from Razmyslov [12]. Unexplained Lie algebra terminology can be found in [1] or [3].

1.5. The following proposition gives new examples of SPI-algebras.

PROPOSITION. *Let G be an arbitrary commutative ring k. Denote by* $\mathbf{K} = \mathrm{qvar}(G)$ *the least quasi variety of Lie algebras containing G. If $L \in \mathbf{K}$, then L is* SPI. *In particular, any free algebra of* $\mathbf{V} = \mathrm{var}(G)$, *the variety generated by G, is* SPI.

Proof. One of the equivalent definitions of a quasi variety [10, Chap. 5] says that this is a nonempty class closed under taking subalgebras and filtered products. In our case it is sufficient to take subalgebras of filtered powers of G. But if G is a subalgebra of an associative PI-algebra A then a filtered power of G is a subalgebra of a filtered power of A with respect to the same filter, that is, a subalgebra of an associative PI-algebra. The last statement of the proposition follows from Birkhoff's description of varieties (see, e.g., [3]). ∎

1.6. The same argument as above and an equivalent description of quasi varieties as locally and multiplicatively closed classes [10, Chap. 5] show that a "local theorem" holds for SPI-algebras:

COROLLARY. *Suppose that every finitely generated subalgebra of a Lie algebra L can be embedded in an associative algebra satisfying a fixed identity. Then L is an* SPI-*algebra.*

1.7. Given a Lie algebra G denote by $\mathrm{Ad}(G)$ the associative subalgebra of $\mathrm{End}_k(G)$ generated by adx, $x \in G$. We do not know whether G being SPI implies that $\mathrm{Ad}(G)$ is PI. But we can prove the following.

PROPOSITION. *Let G be a Lie algebra such that* $\mathrm{Ad}(G)$ *is* PI *and* $H \in \mathrm{var}(G)$. *Then* $\mathrm{Ad}(H)$ *satisfies all the identities of* $\mathrm{Ad}(G)$.

Proof. Let $H = L/M$, where L is a free algebra in $\mathrm{var}(G)$, the Lie algebra variety generated by G. Then L is a subalgebra of G^I for some index set I. Now if $a \in \mathrm{Ad}(L)$, then

$$a = \sum_{i_1,\ldots,i_t} adx_{i_1} \cdots adx_{i_t},$$

where $x_{i_j} \in L$. But each $x_{i_j} = (\alpha_{i_j}^{\alpha})$, $\alpha \in I$, and

$$a^{\alpha} = \sum_{i_1,\ldots,i_t} adx_{i_1}^{\alpha} \cdots adx_{i_t}^{\alpha}$$

is an element of Ad(G). For every polynomial $f(y_1,\ldots, y_p)$ we see that for any $a_1,\ldots, a_p \in L$

$$[f(a_1,\ldots, a_p)(z)]^\alpha = f(a_1^\alpha,\ldots, a_p^\alpha)(z^\alpha)$$

and if $f=0$ holds identically in Ad(G) then the same is true for Ad(L). Now if $\bar{a}_1,\ldots, \bar{a}_p \in \mathrm{Ad}(H)$, then the computation of $f(\bar{a}_1,\ldots, \bar{a}_p)(\bar{z})$ can first be performed in L and then the result "descended" in Ad(H). ∎

Since every centreless Lie algebra L is a subalgebra of Ad(L) we see that every centreless algebra from every variety generated by a finite dimensional algebra is SPI. Razmyslov [13] characterized finitely generated algebras in such varieties as algebras L with soluble ideal M such that $[L, M]$ is nilpotent and L/M is a subdirect product of finite dimensional simple algebras of bounded dimension (when char $k=0$).

1.8. Using the preceding remarks we can prove the following simple result.

PROPOSITION. *Every simple* SPI-*algebra of degree* n (*that is, a subalgebra of an associative* PI-*algebra satisfying an identity of degree* n) *has dimension* $\leqslant n^2/4$ *over its centroid.*

Proof. By Amitsur's theorem [11, p. 68], L is a subalgebra of an associative PI-algebra A with a nil ideal N such that A/N is an algebra from var(M_t), the variety generated by identities of the matrix algebra M_t of order $t \leqslant [n/2]$. By Levitzki's theorem [11, p. 128] every finitely generated subalgebra of N is locally nilpotent. Since no nonabelian simple Lie algebra can be locally nilpotent (see, e.g., [4]) we see that $L \cap N = 0$.

Thus $L \in \mathrm{var}(M_t)$. By 1.8 this implies that $B = \mathrm{Ad}(L)$ is a PI-algebra, more precisely, an algebra from var(M_{t^2}). Since L is an irreducible Ad(L)-module it must have dimension $\leqslant t^2$ over a subfield Z of $\mathrm{End}_k(L)$ which commutes with Ad(L). Recalling the definition of the centroid (see, e.g., [7]) we immediately see that L has dimension over its centroid at most $t^2 \leqslant n^2/4$ as required. ∎

1.9. One may ask: What Lie algebra identities V guarantee that if L is a Lie algebra satisfying V then L is SPI? It follows from Latyšev's results that if V determines a variety $\mathbf{V}$ which is a subvariety of $\mathbf{N}_c\mathbf{A}$ for some c then $L \in \mathbf{V}$ ensures that L is SPI. No other systems of identities with this property are known. A "standard" identity, that is, one of the form

$$S_n(adx_1,\ldots, adx_n)(y) = 0, \tag{1}$$

is not sufficient. Indeed, the algebra W_1 of derivations of $k[x]$, the polynomial ring in one variable over a field k of characteristic 0, satisfies (1) with $n=5$ (I. N. Sumenkov, Novosibirsk, unpublished). By 1.9 if W_1 is SPI it follows that W_1 is finite dimensional over its centroid, which is not so.

(1.10) We recall, for the sequel, that, given a Lie algebra G, the enveloping algebra $U(G)$ is the quotient algebra of the tensor algebra

$$T(G)=k\cdot 1\oplus G\oplus\cdots\oplus G^{\otimes s}\oplus\cdots,\ G^{\otimes s}=\underbrace{G\otimes\cdots\otimes G}_{s},$$

by the ideal generated by $a\otimes b-b\otimes a-[a,b]$, a, $b\in G$. By the Poincaré–Birkhoff–Witt theorem (PBW-theorem) [5], G is a generating Lie subalgebra of $U(G)$ and, given a totally ordered basis of G, a basis of $U(G)$ is given by the images of 1 and standard monomials $e_1\otimes\cdots\otimes e_s$, $e_1\leqslant\cdots\leqslant e_s$, $s\geqslant 1$, $e_i\in E$, which we denote by $e_1\cdots e_s$.

If H is a subalgebra of G then it generates $U(H)$ in $U(G)$. If H is an ideal of G, $H\lhd G$, then $U(G)/K=U(H)$, where K is the ideal generated by H in $U(G)$.

Every linear representation of G uniquely extends to a representation of $U(G)$ and vice versa. Thus the map $x\mapsto adx$, $x\in G$, extends to the adjoint representation of $U(G)$ in $U(H)$, $H\lhd G$, and the left regular representation of $U(G)$ in $U(G)$ restricts to the left regular representation of G in $U(G)$.

2. More Examples of SPI-Algebras

2.1. The following extends an argument from Latyšev [9]. Let L be a Lie algebra with a nilpotent ideal N such that for some positive c $N^c\neq 0$, $N^{c+1}=0$. Choose a basis E of N such that $E=\bigcup_{i=0}^{c}E_i$ is the disjoint union of bases E_c of N^c and E_i, $0\leqslant i\leqslant c$, where $E_i\cup E_{i+1}\cup\cdots\cup E_c$ is a basis of N^i ($N^0=L$). For $x\in E$ define weight $\mathrm{wt}(x)$ by $\mathrm{wt}(x)=i$ if $x\in E_i$. Let $\preccurlyeq$ be a total ordering on E such that $\mathrm{wt}(x)<\mathrm{wt}(y)$ implies $x\preccurlyeq y$.

By 1.10 a basis of the enveloping algebra $U=U(L)$ is formed by (1) in 1.9 and all "ordered" monomials of the form

$$e_1e_2\cdots e_n,\qquad \text{where}\quad e_1\preccurlyeq e_2\preccurlyeq\cdots e_n,\ n\geqslant 1. \tag{1}$$

If m is an element of the form (1) we put $\mathrm{wt}(m)=\mathrm{wt}(e_1)+\mathrm{wt}(e_2)+\cdots+\mathrm{wt}(e_n)$. If $u=\sum_s\alpha_s m_s$, where $0\neq\alpha_s\in k$ and m_s is of the form (1) we put $\mathrm{wt}(u)=\min_s\{\mathrm{wt}(m_s)\}$. For convenience define $\mathrm{wt}(0)=+\infty$, $\mathrm{wt}(1)=0$.

Since in L $[N^i,N^j]\subseteq N^{i+j}$ for all i, $j=0,1,\ldots,c$, we see that

$\mathrm{wt}([x, y]) \geqslant \mathrm{wt}(x) + \mathrm{wt}(y)$, if $x, y \in L$. If $f_1, f_2, \ldots, f_q$ are elements of E, then using the defining relations of the enveloping algebra of the form $xy = yx + [x, y]$, $x, y \in L$, one easily proves that for an unordered product $f_1 f_2 \cdots f_q$ one has $\mathrm{wt}(f_1 f_2 \cdots f_q) \geqslant \mathrm{wt}(f_1) + \mathrm{wt}(f_2) + \cdots + \mathrm{wt}(f_q)$. Hence we deduce that

$$\mathrm{wt}(uv) \geqslant \mathrm{wt}(u) + \mathrm{wt}(v) \tag{2}$$

for arbitrary $u, v \in U$. Therefore for every $s = 0, 1, \ldots$, the set U_s of all elements u of U with $\mathrm{wt}(u) \geqslant s$ is an ideal of U such that

(i) $U_1^s \subseteq U_s$;

(ii) $U_s \cap L = 0$ for all $s \geqslant c + 1$.

Part (i) is clear from (2). For (ii) it suffices to note that 0 is the only element x of L with $\mathrm{wt}(x) \geqslant c + 1$. This enables us to prove the following.

2.2. THEOREM. *Let k be a field and L is a Lie algebra over k with a nilpotent ideal N such that the enveloping algebra of $G = L/N$ is a* PI-*algebra. Then L is an* SPI-*algebra.*

Proof. By 2.1, L can be embedded in the associative algebra U/U_{c+1}, where U/U_1 is the enveloping algebra of $G = L/N$ and $(U_1/U_{c+1})^{c+1} = 0$. Under the given hypotheses U/U_{c+1} is a PI-algebra. ∎

2.3. When the ground field has characteristic 0, 2.2 gives theorem 4 of Latyšev [9]. If $\mathrm{char}(k) = p > 0$, the description of Lie algebras G for which $U(G)$ is a PI-algebra is given in Bahturin [2]. Using it we get the following.

COROLLARY. *Let* $\mathrm{char}(k) = p > 0$. *Suppose that L is a Lie algebra with a soluble ideal M such that M^2 is nilpotent and* $\dim(L/M) < \infty$. *If the natural action of L on M/M^2 is algebraic of bounded degree, then L is an* SPI-*algebra. In particular, if L has an ideal M of finite codimension with $[L, M]$ nilpotent, then L is* SPI.

2.4. The following examples illustrate Theorem 2.2. Firstly, there exist locally nilpotent algebras which are not SPI. We take k with $\mathrm{char}(k) = 0$ and $L = L_s(\mathbf{AN}_2)$ the free algebra of rank $s > 1$ in the variety of all abelian-by-nilpotent of class two algebras. It is known [3, Chap. 3] that $\bigcap_{n=1}^{\infty} L^n = 0$. Put $P_n = L/L^n$ and consider $P = P_1 \times P_2 \times \cdots \times P_n \times \cdots$. Then P is locally nilpotent but not SPI. For, otherwise, by 1.6 L is SPI and this contradicts 3.2.

Secondly, a locally nilpotent SPI-algebra is not necessarily nilpotent even if it is metabelian, that is, soluble of length 2. Indeed, take $L = L_\infty(\mathbb{A}^2)$,

$Q_n = L/L^n$, $Q = Q_1 \times \cdots \times Q_n \times \cdots$. Then, by 2.2, Q is SPI. Q is locally nilpotent but not nilpotent since it generates the variety $\mathbf{A}^2$ which is not nilpotent.

3. Some Necessary Conditions

3.1. Lemma. *Let L be a Lie algebra over a field k of characteristic zero with a faithful finite dimensional module V which is simple over some extension field K, k. Then every soluble ideal M of L is central in L.*

Proof. See Jacobson [7, Chap. 3]. ∎

3.2. Theorem. *Let k be a field of characteristic* 0 *and M a locally soluble ideal of an* SPI-*algebra L. Then $N = [L, M]$ is locally nilpotent.*

Proof. It is sufficient to restrict the proof to the case where L is finitely generated. Then L is a subalgebra of a finitely generated associative PI-algebra A. The Jacobson radical J of A is a nil algebra [11, p. 102]. A/J is a subdirect product of primitive algebras A_α, $\alpha \in I$, finite dimensional over their centres K_α. Each A_α is generated by the image of L under a homomorphism $\gamma_\alpha\colon A \to A_\alpha$. Therefore $L_\alpha = \gamma_\alpha(L)$ is a Lie algebra with a faithful module (that of A_α) over K_α. By 3.1 the image $M_\alpha = \gamma_\alpha(M)$ is central in L_α, that is, $[L_\alpha, M_\alpha] \subseteq J$ for all $\alpha \in I$. Since the intersection of all $\operatorname{Ker} \gamma_\alpha$ is in J, $N = [L, M] \subseteq J$. Now every finitely generated subalgebra B of N lies in a finitely generated subalgebra of J. Since J is nil, by Levitzki's theorem [11, p. 128] B is nilpotent. A standard argument shows that if B is nilpotent of class c, then N is nilpotent of class $2c - 1$ as Lie algebra. ∎

3.3. Corollary. (i) *Let R, N be respectively the locally soluble and the locally nilpotent radicals of an* SPI-*algebra L over a field k of characteristic* 0. *Then $[L, R] \subseteq N$.*

(ii) *Let k be a field of characteristic* 0, *L an* SPI-*algebra over k, G a soluble subalgebra of L, x an element of G^2. Then $\operatorname{ad} x$ is nilpotent.* ∎

What we actually require in 3.2 is that all finite dimensional homomorphic images of M are soluble.

3.4. Theorem. *Let k be a field of characteristic* 0 *and L an* SPI-*algebra over k. If L is soluble then L^2 is locally nilpotent. If L is finitely generated (virtually) soluble, then it is (virtually) nilpotent-by-abelian. There exist soluble* SPI-*algebras L with nonnilpotent L^2.*

Proof. The first statement follows directly from 3.2. The third will be proved in 4.5 when we will have more technique available. In the proof of the second we will be using the following corollary of Širšov's Height Theorem [14] due to Razmyslov [12, 13].

3.5. LEMMA. *Let J be the Jacobson radical of a finitely generated associative* PI-*algebra A. If J is a finitely generated ideal of A, then it is nilpotent.*

3.6. We continue the proof of 3.4. So far, L is a Lie subalgebra of an associative PI-algebra A. Since L is finitely generated, the same is true for A. By the hypothesis L has a soluble ideal M with $\dim(L/M) < \infty$. The same argument as in 3.2 shows that $N = [L, M] \subseteq J$, the Jacobson radical of A. L/N is an extension of $M/[L, M]$ by L/M. Since both algebras are finite dimensional, L/N is finite dimensional and N is a finitely generated ideal of L. Let I be the two-sided ideal of A generated by N. Since A is generated by L, I is a finitely generated two-sided ideal of A. The ideal J/I is an ideal of the associative algebra A/I which is a homomorphic image of the enveloping algebra U of L/N. By [7, Chap. 5], U is Noetherian and thus J/I is finitely generated as an ideal of A/I. Finally, J is a finitely generated ideal of A. Applying 3.5 we see that, as required, $N = [L, M]$ is nilpotent. If L is soluble, then $M = L$ and therefore L^2 is nilpotent. ∎

3.7. COROLLARY. *A homomorphic image of a finitely generated soluble* SPI-*algebra of characteristic zero is* SPI.

3.8. The remaining part of this section is occupied by the following.

THEOREM. *Let k be a field of characteristic* 0 *and L a finitely generated virtually soluble* SPI-*algebra. Then L is a split extension of a semisimple finite dimensional subalgebra G and a soluble ideal M such that every chief factor of M as a natural G-module is of finite bounded dimension.*

Proof. Let M be a soluble ideal of L such that $\mathfrak{g} = L/M$ is finite dimensional semisimple. By 3.4, $N = [L, M]$ is nilpotent and L/N is finite dimensional. Applying Levi–Malcev theorem [6, Chap. 1], L/N is a direct product

$$L/N = G/N \times M/N,$$

where $G/N = \mathfrak{g}$ is semisimple and M/N is central. $\mathfrak{g}$ is naturally represented by endomorphisms of the spaces N^t/N^{t+1}, $t = 1,..., c$, where $N^c \neq 0$, $N^{c+1} = 0$. To see this define δ_t, $t = 1,..., c$, as follows. Let $x \in \mathfrak{g}$, $g + N$ a corresponding coset of G/N ($g \in G$), $y = n + N^{t-1}$ an element of N^t/N^{t+1} ($n \in N^t$). Put

$$[\delta_t(x)](y) = [g, n] + N^{t+1}. \tag{1}$$

Since N centralizes each N^t/N^{t-1} this definition is correct.

3.9. Lemma. *Let g be a split semisimple finite dimensional Lie algebra over a field k of characteristic 0. Denote by $\mathbf{h}$ a splitting Cartan subalgebra of $\mathbf{g}$, R the corresponding root system, B a basis of R. Let $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ be the corresponding triangular decomposition. Then for every positive integer l there exists only finitely many nonisomorphic irreducible representations β of $\mathbf{g}$ such that for every $x \in \mathbf{n}_- \cup \mathbf{n}_+$ one has $[\beta(x)]^l = 0$.*

(All the unexplained notation can be found in [6, Chap. 1].)

Proof. Let V be a module of β as above, v a nonzero element of V, e_α an element of $\mathbf{n}_+$ ($\alpha \in R_+$). For some $n \leqslant 1$, $e_\alpha^n v = 0$, $e_\alpha^{n-1} v \neq 0$. Denote by V_α the subspace of V generated by all the elements in V of the form

$$h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_t} e_{\beta_1} \cdots e_{\beta_s} e_\alpha^{n-1} v, \tag{2}$$

where $h_{\alpha_j} \in \mathbf{h}$, $e_{\beta_i} \in \mathbf{n}_+$, α_j, $\beta_i \neq \alpha$. V_α is a nonzero $\mathbf{g}$-submodule of V. To verify this it suffices to show that if w is an element of the form (2), then $h_\alpha w \in V_\alpha$. But $[h_\alpha, h_\beta] = 0$ and $[h_\alpha, e_\beta] = \gamma_\beta e_\beta$, where $\gamma_\beta \in k$. To compute $h_\alpha e_\alpha^{n-1}$ note that

$$\begin{aligned} 0 &= [e_\alpha^n, e_{-\alpha}](v) \\ &= \left(\sum_{p=0}^{n-1} e_\alpha^p [e_\alpha, e_{-\alpha}] e_\alpha^{n-1-p} \right)(v) = [2h_\alpha e_\alpha^{n-1} - 2n(n-1) e_\alpha^{n-1}](v). \end{aligned}$$

This shows that $(h_\alpha e_\alpha^{n-1})(v) = (n-1)e_\alpha^{n-1}(v)$ and, with the above, that $h_\alpha w \in V_\alpha$.

Since V is simple, $V = V_\alpha$. We see that every element of V is a linear combination of elements of the form (2) not involving h_α. Repeating the same procedure for other basic roots we get rid of other elements in $\mathbf{h}$ and, finally, represent all elements of V as linear combinations of elements of the form (2) with $t = 0$. By the hypothesis in this case V is finite dimensional. Moreover, $\dim(V)$ is bounded by l Card(R_+). The finiteness of the number of irreducible representations of bounded dimension is left as an exercise [7, p. 236]. ∎

3.10. We continue the proof of 3.8, initially restricting it to the case where $\mathbf{g} = L/M$ is split semisimple. Let B denote a finitely generated subalgebra of L such that $B+N/N=\mathbf{h}\oplus\mathbf{n}_+$. B is SPI and, therefore, 3.5 applies. There exists a positive integer d such that any associative product of at least d elements in B^2 is zero. Similarly there exists n such that any product of at least n elements of an associative ideal C of A generated by N is zero. Now let $e_\alpha = x_\alpha + N$, $h_\alpha = y_\alpha + N$, $x_\alpha, y_\alpha \in B$. Then

$$x_\alpha + N = e_\alpha = 2[e_\alpha, h_\alpha] = 2[x_\alpha, y_\alpha] + N,$$

that is,

$$x_\alpha = 2[x_\alpha, y_\alpha] + p, \ p \in N.$$

Let us compute $x_\alpha^{d+n-1} = (2[x_\alpha, y_\alpha]+p)^{d+n-1}$. The right-hand side is a linear combination of the monomials of the form

$$[x_\alpha, y_\alpha]^s p_1 p_2 \cdots p_{d+n-1-s},$$

where $0 \leqslant s \leqslant d+n-1$ and $p_1, p_2,..., p_{d+n-1-s} \in C$. As usual, this gives the desired result. Since $\mathbf{g}$ is finite dimensional, proceeding in the same way gives a system of elements x_α, $\alpha \in R_+ \cup R_-$ and a positive integer l' such that $e_\alpha = x_\alpha + N$ and $x_\alpha^{\alpha'} = 0$. Recalling the definition (1) of δ_t we derive the existence of 1 with $[\delta_t(x)]^l = 0$ for all $x \in n_- \cup n_+$. By 3.9 there exists only finitely many irreducible representations $\beta_1,..., \beta_q$ such that every chief factor of N^t/N^{t+1} is isomorphic to the module of one of $\beta_1,..., \beta_q$. Therefore only finitely many ideals $P_1,..., P_s$ of $U(\mathbf{g})$ may be annihilators of irreducible modules occuring in δ_t, $t = 1,..., c$, and $\dim(U(\mathbf{g})/P_i) < \infty$.

Now if L is such that $\mathbf{g} = L/M$ is not necessarily split, there exists a (finite) extension K of k such that $\mathbf{g}_K = \mathbf{g} \otimes_k K$ is split. Obviously $\mathbf{g}_K = L_K/M_K$. Since $[X, Y]_K = [X_K, Y_K]$, X, $Y \subseteq L$, all the preceding argument applies to L_K. Therefore, only finitely many ideals $Q_1,..., Q_r$ of $U(\mathbf{g}_K) = U(\mathbf{g})_K$ may be annihilators of chief factors of $\mathbf{g}_K$-modules $N_K^t/N_K^{t+1} = (N^t/N^{t+1})_K$. Assume that P is an annihilator of a chief factor S in N^t/N^{t+1} in $U(\mathbf{g})$. Then P_K is the annihilator of S_K in $U(\mathbf{g}_K)$. By [6, 1.2.19], S_K is a direct sum of finitely many irreducible $\mathbf{g}_K$-submodules. Therefore $P_K = Q_{i_1} \cap \cdots \cap Q_{i_s}$, $1 \leqslant i_1,..., i_s \leqslant r$. The finiteness of the number of P's is now obvious.

3.11. Lemma. *Let* $\mathbf{g}$ *be a finite dimensional semisimple Lie algebra and* V *a* $\mathbf{g}$*-module. Suppose that the annihilators of all chief factors of* V *belong to a finite set* $P_1,..., P_s$ *of ideals of* $U(\mathbf{g})$ *such that* $\dim[U(\mathbf{g})/P_i] < \infty$. *Then* V *is locally finite.*

Proof. It is sufficient to restrict the proof to the case where V is cyclic. Since $U(\mathbf{g})$ is noetherian, the same is true for V. Let T be an arbitrary submodule of V such that $\dim(V/T) < \infty$. Since $\mathbf{g}$ is semisimple V/T is a direct sum of modules with annihilators in the set $\{P_1,\ldots, P_s\}$. Therefore if $I = \mathrm{Ann}_{U(\mathbf{g})}(V)$, for some integer m, $\dim[U(\mathbf{g})/I] < m$. This shows that $\dim(V/T)$ is bounded by m. Let T_0 be such that $\dim(V/T_0)$ is the largest possible finite. Since V is noetherian, $T_0 \neq V$ and T_0 is finitely generated. If T_0 is not trivial, then the same argument applies to a cyclic quotient of T_0 and gives a submodule T_1 of T_0 with $\infty > \dim(V/T_1) > \dim(V/T_0)$. Thus T_0 is trivial and the proof is complete. ∎

3.12. To complete the proof of 3.8 we must show that L is a split extension of a subalgebra $G = \mathbf{g}$ and M. Induction on c, the nilpotency class of N. (We are not using that L/N^t is SPI, but merely that N^t/N^{t+1} is a locally finite g-module, 3.11.) Since L/N is split we have the basis for our induction.

Proceeding by induction assume that $\bar{G}$ is a subalgebra of L such that $\bar{G} \supseteq N^c$ and $\bar{G}/N^c \cong \mathbf{g}$. Let $e_1,\ldots, e_n$ be a basis of $\bar{G}$ over N^c. Then $[e_i, e_j] = \sum_t \gamma_{ij}^t e_t + u_{ij}$, where $\gamma_{ij}^t \in k$ and $u_{ij} \in N^c$. Let V be the submodule of the g-module N^c generated by u_{ij}, $1 \leqslant i, j \leqslant n$. By 3.11, V is finite dimensional. Denote by $\tilde{G}$ a subalgebra of L generated by $e_1,\ldots, e_n$. Obviously $\tilde{G} \cap N^c = V$ and $\tilde{G}/V = \tilde{G} + N^c/N^c = \bar{G}/N^c = \mathbf{g}$. Therefore, $\tilde{G}$ is finite dimensional. By Levi–Malcev theorem there exists a subalgebra G of $\tilde{G}$ isomorphic to $\mathbf{g}$. Since G is semisimple, $G \cap M = 0$ and, consequently, L is a split extension of G and M. Every chief factor S of M as G-module is isomorphic to a chief factor of some N^t/N^{t+1} or M/N as g-module. Combining of 3.9 and 3.11 gives that $\dim_k(S)$ is finite bounded. ∎

4. Some Sufficient Conditions

4.1. The following lemma can be found in various books; e.g., see [6, Chap. 1].

Lemma. *Let L be a Lie algebra of the form $L = G \oplus M$, where G is a subalgebra and M is an ideal. Let $\pi: M \to \mathrm{End}_k U(M)$ be defined by $\pi,(m)u = mu$, and $\varphi: G \to \mathrm{End}_k U(M)$ be the canonical extension of the map $\varphi: G \to \mathrm{End}_k(M)$ such that $\varphi(g)m = [g, m]$. Then the linear map $\psi: L \to \mathrm{End}_k U(M)$ with $\psi(g+m) = \varphi(g) + \pi(m)$ is a homomorphism.*

Proof. Let $l(x)$, $r(x)$ be as in 1.2. We have to prove that

$$\psi([x, y]) = [\psi(x), \gamma(y)].$$

Since π, φ are homomorphisms we may take $x \in G$, $y \in M$. Then, since $[x, y] \in M$,

$$\varphi([x, y]) = \pi([x, y])$$

and

$$\begin{aligned}[\psi(x), \psi(y)] &= [\varphi(x), \pi(y)] = [l(x) - r(x), l(y)] \\ &= [l(x), l(y)] = l([x, y]) = \pi([x, y]),\end{aligned}$$

completing the proof. ∎

4.2. Theorem. *Let $L = G \oplus M$, where G is an* SPI-*algebra and M an ideal of L such that $N = [L, M]$ is nilpotent. Denote by V the annihilator of the adjoint G-module N in $U(G)$. If $U(G)/V$ is a* PI-*algebra then L is an* SPI-*algebra.*

Proof. Since G is SPI, there exists a homomorphism α of L into a PI-algebra A such that $\operatorname{Ker} \alpha = M$. Suppose we have constructed a homomorphism β into another PI-algebra B with $\operatorname{Ker} \beta \cap M = 0$. Define $\gamma: L \to A \times B$ by $\gamma(l) = (\alpha(l), \beta(l))$. γ is a monomorphism of L into a PI-algebra, proving the result.

To construct β we first proceed as in 2.1. Denote by W the intersection of U_{c+1} with $U = U(M)$ (c the nilpotent class of N). Let π, φ, and ψ be as in 4.1. Since both π and φ respect W the same is true for ψ, so we have a homomorphism $\bar{\psi}$ of L into $\operatorname{End}_k(U/W)$. We now prove that $\bar{\psi}(U(L))$ is a PI-algebra.

4.3. Lemma. *Let G be a Lie algebra and N a submodule of a G-module M such that $GM \subseteq N$. Let T_c be the subspace of the tensor algebra $T = T(M) = \oplus_{t=0}^{\infty} M^{\oplus t}$ generated by all tensors of the form*

$$m_1 \otimes m_2 \otimes \cdots \otimes m_r,$$

where $m_i \in M$ and at least $c + 1$ of the m_i's are in M. Put $Q = U(G)$ and assume that $Q/\operatorname{Ann}_Q(N)$ is a PI-*algebra. Then $Q/\operatorname{Ann}_Q(T/T_c)$ is a* PI-*algebra, too.*

Proof. From the definition [5] the G-structure on

$$M^{\otimes t} = \underbrace{M \otimes \cdots \otimes M}_{t}$$

is, in fact, a composition of the natural $U(G)^{\otimes t}$-module structure on $M^{\otimes t}$, that is, for $u_1,..., u_t \in U(G)$ and $m_1,..., m_t \in M$

$$(u_1 \otimes \cdots \otimes u_t)(m_1 \otimes \cdots \otimes m_t) = (u_1 m_1) \otimes \cdots \otimes (u_t m_t) \tag{1}$$

and the homomorphism $\Delta_t: G \to U(G)^{\otimes t}$ is given by

$$\Delta_t(g) = g \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes g \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes g.$$

Therefore, it is sufficient to prove that there exists a polynomial identity satisfied by the image of every $U(G)^{\otimes t}$ in the algebra $\mathrm{End}_k(M^{\otimes t}/T_c \cap M^{\otimes t})$.

Now since $Q/\mathrm{Ann}_Q(N)$ is a PI-algebra, $Q/\mathrm{Ann}_Q(M)$ is also PI. Clearly $Q^{\otimes t}/\mathrm{Ann}_{Q^{\otimes t}}(M^{\otimes t})$ is a homomorphic image of $[Q/\mathrm{Ann}_Q(M)]]^{\otimes t}$. For, if $(\bar{q}_1,..., \bar{q}_t) \in (Q/\mathrm{Ann}_Q(M),..., Q/\mathrm{Ann}_Q(M))$ we may define a mapping

$$(\bar{q}_1,..., \bar{q}_t) \mapsto q_1 \otimes \cdots \otimes q_t + \mathrm{Ann}_{Q^{\otimes}}(M^{\otimes t}). \tag{2}$$

If $q_1 = q_1' + p$, where $p \in \mathrm{Ann}_Q(M)$ then $q_1 \otimes \cdots \otimes q_t - q_1' \otimes \cdots \otimes q_t = p \otimes \cdots \otimes q_t$ and this element annihilates $M^{\otimes t}$. By the universality property of tensor products, (2) extends to an epimorphism. Since $Q/\mathrm{Ann}_Q(M)$ is a PI-algebra, by Regev's theorem the same is true for each tensor product $[Q/\mathrm{Ann}_Q(M)]^{\otimes t}$ and so for each of the algebras $Q^{\otimes t}/\mathrm{Ann}_{Q^{\otimes t}}(M^{\otimes t})$. Let $w(x_1,..., x_n) \equiv 0$ be a multilinear identity holding in all these algebras for $t = 1,..., c$. We will prove that $w(x_1,..., x_n) \equiv 0$ holds in the images of all the $Q^{\otimes t}$ in $\mathrm{End}_k(M^{\otimes t}/T_c \cap M^{\otimes t})$. We may assume, of course, that $t \geqslant c + 1$. From (1) it is obvious that the image of $Q^{\otimes t}$ is the sum of the images of the subalgebras $Q_{i_1,...,i_c}$ spanned by the tensors $u_1 \otimes \cdots u_t$ in which the only terms distinct from 1 are in the places $i_1,..., i_c$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_c \leqslant t$. Each of these subalgebras is isomorphic to $Q^{\otimes c}$ and acts on $M^{\otimes t}$ in the same way as $Q^{\otimes c}$ acts on $M^{\otimes c}$. Now if $(i_1,..., i_c) \neq (j_1,..., j_c)$ and $u \in Q_{i_1,...,i_c} - Q_{j_1,...,j_c}$, $v \in Q_{j_1,...,j_c} - Q_{i_1,...,i_c}$ then uv acts on $M^{\otimes t}/T_c \cap M^{\otimes t}$ trivially. This shows that in computing the value of $w(x_1,..., x_n)$ in the image of $Q^{\otimes t}$ we can restrict to elements from some fixed $Q_{i_1,...,i_c}$. Therefore $w(x_1,..., x_n) = 0$ holds identically in all images of $Q^{\otimes t}$. Finally, since the G-module T is the direct sum of G-modules $M^{\otimes t}$ we see that $Q/\mathrm{Ann}_Q(T/T_c)$ is a PI-algebra. ∎

4.4. We continue the proof of 4.2. By definition [5], $U(M)$ as a G-module is a quotient module of $T(M)$ and the images of the elements of T_c have weight $\geqslant c + 1$ in $U(L)$ (we adopt the notation of 2.1 again). Therefore, in fact, U/W is a quotient module of T/T_c, and, denoting by S the annihilator of U/W in $Q = U(G)$ we see that Q/S is a PI-algebra. Let $w(x_1,..., x_n) \equiv 0$ be a multilinear identity holding in Q/S. We will show that

$$w(x_1,..., x_n)^{c+1} = 0 \tag{3}$$

holds identically in $\bar{\psi}(U(L))$. Remark first of all that if P is an ideal generated by N in $U(L)$ then $\bar{\psi}(P^{c+1})=0$. Indeed, every element of P^{c+1} has weight at least $c+1$ and is a linear combination of monomials

$$e_{i_1}\cdots e_{i_t}e_{j_1}\cdots e_{j_r}, \qquad \text{where } e_{i_1}\cdots e_{i_t}\in E \text{ and } e_{j_1}\cdots e_{j_r}\in E_1\cup\cdots\cup E_c. \quad (4)$$

We may write $E_0=E_0'\cup E_0''$, where E_0' is a basis of G and $E_0''\cup E_1\cup\cdots\cup E_c$ is a basis of M so that, for some s, $e_{i_1},\dots,e_{i_s}\in E'$, $e_{i_{s+1}},\dots,e_{i_t}\in E_0''$. Applying $\bar{\psi}$ to (4) we get

$$\bar{\psi}(e_{i_1}\cdots e_{j_r})(m)=[e_{i_1},\dots,e_{i_s},e_{i_{s+1}}\cdots e_{j_r}m]$$

and the right-hand side has weight $\geqslant c+1$ (not $c+2$ because, possibly, $m=1$) and so belongs to W.

Now $\bar{\psi}(U(L))$ is an extension of $\bar{\psi}(P)$ by $\bar{\psi}(U(L))/\bar{\psi}(P)$. This latter, however, is the image of $U(L)/P=U(L/N)$, the enveloping algebra of $G\times M/[L,M]$. The enveloping algebra of the direct product is the tensor product $U(G)\otimes U(M/[L,M])$. Since the image $\bar{\psi}(U(G))$ satisfies the multilinear identity $w(x_1,\dots,x_n)\equiv 0$ and since $U(M/[L,M])$ is commutative, $w(x_1,\dots,x_n)$ holds in $\bar{\psi}(U(L))/\bar{\psi}(P)$. Thus (3) holds in $\bar{\psi}(U(L))$ and the proof is complete. ∎

4.5. Now we can complete the proof of Theorem 3.4. For this let k be a field of characteristic $\neq 2$ and let L be of the form $L=G\oplus M$, where G is a free associative anticommutative algebra on $x_1,\dots,x_n,\dots$, and M the left regular G-module, that is, the same vector space as G with the multiplication by the elements of G on the left. G is a Lie algebra with respect to $[x,y]=xy-yx$, and M is an abelian Lie algebra. A basis of G over k is formed by $x_{i_1}x_{i_2}\cdots x_{i_n}$, $i_1<i_2<\cdots<i_n$, $n\geqslant 1$. To avoid unnecessary ambiguity we denote the corresponding elements of M by y's. So, the commutator of two elements $x_{i_1}\cdots x_{i_m}$ from G and $y_{j_1}\cdots y_{j_n}$ from M is given by

$$[x_{i_1}\cdots x_{i_m},y_{j_1}\cdots y_{j_n}]=y_{i_1}\cdots y_{i_m}y_{j_1}\cdots y_{j_n}. \qquad (5)$$

Clearly M is a Lie G-module if G is considered as a Lie algebra, too. So, L is a Lie algebra. It is easy to see (and is well known, e.g., [8, p. 260]) that G satisfies $[[x,y],z]\equiv 0$, that is, is nilpotent of class 2 as a Lie algebra. So, L satisfies

$$[[x_1,y_1,z_1],[x_2,y_2,z_2]]\equiv 0,$$

that is, L is a soluble Lie algebra. The derived algebra L^2 of L is spanned by

$x_{i_1}x_{i_2}\cdots x_{i_{2t}}$ and $y_{j_1}y_{j_2}\cdots y_{j_r}$, where t, $r \geqslant 1$. Therefore for every $n>1$, the following commutator of elements from L^2 is nonzero:

$$[x_1x_2,\, x_3x_4, \ldots, x_{2n-1}x_{2n},\, y_{2n+1}] = y_1 \cdots y_{2n+1}.$$

Therefore L is not nilpotent-by-abelian. Now G is SPI, M is nilpotent, and $U(G)/\mathrm{Ann}_{U(G)}M$ is a PI-algebra, since the action of the Lie algebra G is, in fact, the action of the associative PI-algebra G. Therefore, we can apply 4.2 and thus L is an example of a soluble SPI-algebra with nonnilpotent derived algebra and 3.4 is complete. ∎

4.6. In this section we complete our description of virtually soluble SPI-algebras.

THEOREM. *Let k be a field of characteristic zero. A finitely generated virtually soluble Lie algebra L is an* SPI-*algebra if and only if it is a semidirect product of a finite dimensional semisimple algebra G with a soluble ideal M such that $N=[L,M]$ is nilpotent and all the chief factors of M as G-module are of finite bounded dimension. M has finitely many generators as Lie algebra.*

Proof. The necessity of all conditions was proved in 3.8. The only thing remaining is to show that M is finitely generated. To prove this note that a finite system X of generators for L can be chosen in the form $X=Y\cup Z$, where Y is a basis of G and Z is a basis of M. By 3.11, M is locally finite. An easy exercise involving the *PBW*-theorem shows that M is generated by a basis Z' of the finite dimensional G-submodule of M generated by Z.

To prove that the conditions are sufficient we use 4.2. By the same argument as in 3.10, if $I=\mathrm{Ann}_{U(G)}(N)$, then $\dim[U(G)/I]$ is finite and $U(G)/\mathrm{Ann}_{U(G)}(N)$ is a PI-algebra. Now 4.2 applies and the proof is complete. ∎

4.7. Finally we give an example of a Lie algebra satisfying the hypothesis of theorem 4.6.

Let $G=\mathrm{sl}(2,k)$, $N=\langle x_i, y_i, z\ (i\in Z) \mid [x_i,y_j]=\delta_{ij}z,\ [x_i,x_j]=[y_i,y_j]=[x_i,z]=[y_i,z]=0\rangle$. Put $M=kd\oplus N$ with $[d,z]=0$, $[d,x_i]=x_{i+1}-x_{i-1}$, $[d,y_i]=y_{i+1}-y_{i-1}$. It is proved in [3, Chap. 4] that M is a centre-by-metabelian Lie algebra. Define L as a semidirect product of G and M with respect to the following action of g on M. $Gd=Gz=0$ and N is the direct sum of canonical 2-dimensional $\mathrm{sl}(2,k)$-modules $N_i=kx_i\oplus ky_i$, $i\in Z$. It is easy to verify that L is a Lie algebra and that it does not reduce to any simpler classes of Lie algebras.

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